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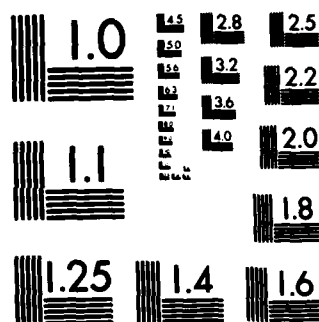
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RECENT DEVELOPMENTS AND OPEN
PROBLEMS IN THE MATHEMATICAL
THEORY OF VISCOELASTICITY

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ABSTRACT

45 This paper presents a nontechnical review of current research efforts in the mathematical theory of viscoelastic materials. Recent results concerning the existence, uniqueness and regularity of solutions for initial value problems as well as for steady flow problems are discussed and a number of open problems are pointed out. *Originator-supplied Keywords include:*

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SIGNIFICANCE AND EXPLANATION

This paper is based on a lecture presented at the Symposium on Viscoelasticity and Rheology, Madison, October 1984. The lecture was aimed at explaining to a general audience some of the mathematical problems associated with model equations for viscoelastic materials and giving an impression of the current state of efforts in this field. Questions of existence, uniqueness and regularity of solutions to initial value problems and to steady flow problems are discussed and some open problems, which the author considers to be of particular interest, are pointed out.

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RECENT DEVELOPMENTS AND OPEN PROBLEMS IN THE MATHEMATICAL THEORY OF VISCOELASTICITY

Michael Renardy

1. EXISTENCE RESULTS FOR INITIAL VALUE PROBLEMS

The existence of solutions for systems of ordinary differential equations

$$\dot{u} = f(u), u(0) = u_0, \quad (1.1)$$

can be established by using the iteration

$$\dot{u}^{n+1} = f(u^n), u^{n+1}(0) = u_0, \quad (1.2)$$

which can be shown to converge for small enough $t > 0$. For partial differential equations such as

$$u_t = \phi(u_x)_x, u(t = 0, x) = u_0(x), \quad (1.3)$$

the iteration (1.2) cannot be used. This is because at each step of the iteration we would be taking two x -derivatives and, if data are sufficiently oscillatory, these x -derivatives will grow larger and larger. An iteration that can be used is the following

$$u_t^{n+1} = \phi'(u_x^n) u_{xx}^{n+1}, u^{n+1}(t = 0, x) = u_0(x). \quad (1.4)$$

This example illustrates a general strategy used for so-called "quasilinear" equations. In such equations the highest derivatives occur only linearly, and nonlinearities contain only lower-order derivatives. One then puts an $n + 1$ on the highest derivatives (the "principal terms" in the equation) and an n on lower order derivatives. Two things must then be proved: first that the linear problems to be solved at each step are well-posed and secondly, that the iteration converges.

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In order to apply these ideas to problems in viscoelasticity, we have to know that the equations can be written in some quasilinear form and we have to know what the principal terms are. In the following, we illustrate this for a class of one-dimensional model equations, whose structure is typical of problems arising in viscoelasticity. Let us consider equations of the form

$$u_{tt} = \mu f(u_x, u_{xt})_x + g(u_x)_x + \int_{-\infty}^t m(t - \tau) h(u_x(t), u_x(\tau))_x d\tau \quad (1.5)$$

with appropriately smooth boundary and initial data. For $\mu > 0$, the term of highest differential order on the right hand side is the one involving u_{xxt} . If the derivative of f with respect to the second argument is positive, (1.5) can be regarded as a parabolic equation and the iteration to be used is

$$u_{tt}^{n+1} = \mu D_2 f(u_x^n, u_{xt}^n) u_{xxt}^{n+1} + \mu D_1 f(u_x^n, u_{xt}^n) u_{xx}^n + g(u_x^n)_x + \int_{-\infty}^t m(t - \tau) h(u_x^n(t), u_x^n(\tau))_x d\tau. \quad (1.6)$$

Under appropriate smoothness assumptions it can be shown that such a scheme is well-defined and converges. This is accomplished by adapting a result due to Sobolevskii [36] for equations without integral terms. At each step of the iteration, we have to solve a linear parabolic equation with time-dependent coefficients. In Sobolevskii's approach, this equation is regarded as a perturbation of the problem with time-independent coefficients, evaluated at the initial time $t = 0$, and a convergent iteration based on the variation of constants formula can be formulated. Finally, it is proved that the mapping $u^n \rightarrow u^{n+1}$ given by (1.6) is a contraction in an appropriate Banach space.

One-dimensional model problems of parabolic type are discussed e.g. in [11], [29]. In [30], the same ideas are extended to three-dimensional flows of incompressible fluids under both displacement and traction boundary conditions.

If $u = 0$, m , g and h are smooth, and m , g' and $D_1 h$ are positive, then (1.5) can be regarded as a hyperbolic equation. The appropriate iteration method is

$$u_{tt}^{n+1} = g'(u_x^n) u_{xx}^{n+1} + \int_{-\infty}^t m(t - \tau) D_1 h(u_x^n(t), u_x^n(\tau)) u_{xx}^{n+1}(\tau) + D_2 h(u_x^n(t), u_x^n(\tau)) u_{xx}^n(\tau) d\tau. \quad (1.7)$$

It looks surprising at first that we can put an n on the $u_{xx}(\tau)$ -term, since this term appears to be of the same differential order as the $u_{xx}(t)$ -term. However, when we differentiate the equation with respect to t , then $u_{xx}(t)$ becomes u_{xxt} , but $u_{xx}(\tau)$ remains $u_{xx}(\tau)$. This explains, on a heuristic level, why this term can be regarded as being of lower order.

At each step of the iteration, one now has to solve a linear hyperbolic equation with time-dependent coefficients. Existence of solutions for such problems can be shown e.g. by the implicit Euler scheme. Again, a contraction argument is used to show convergence of the iteration. This argument consists essentially of two steps: First, it is shown that the iterates remain bounded in a certain norm (e.g. that all third derivatives of u^n have bounds independent of n). Then one uses this fact to show that the iterates converge in a weaker norm (i.e. one proves that second, rather than third, derivatives of u^n converge). An abstract formulation of these ideas is given in the work of Kato [17], [19], [20].

Applications to problems in viscoelasticity are due to Mac Camy [23], Staffans [37], Dafermos and Nohel [5], [6], Hrusa [13], Kim [21], Heard [12] and the author [31], [32]. Three-dimensional problems with incompressibility have been

studied [21], [32], but only for the pure Cauchy problem posed on all of space, not for initial-boundary value problems.

There is some interest in relaxing the assumption that m is smooth. The first term on the right hand side of (1.5) can be thought of as arising from setting m equal to the derivative of the δ -function. It is only reasonable to think that m may also have weaker singularities as $t - \tau \rightarrow 0$. If m is unbounded at 0, the iteration (1.7) will not work. Instead, one will have to put $n + 1$ on the $u_{xx}(\tau)$ -term as well. This means that the "quasilinearized" problems to be solved at each iteration are now history dependent. (In both the parabolic and hyperbolic cases the linear problems involved no history dependence at all, i.e. history dependence was basically treated as a perturbation.) In a recent paper [15], W. Hrusa and the author show that such an iteration converges for the special problem

$$u_{tt} = \phi(u_x)_x - \int_{-\infty}^t m(t - \tau) \psi(u_x(\tau))_x d\tau, \quad (1.8)$$

where ϕ' , ψ' and $\phi' - \psi' \int_0^\infty m(\tau) d\tau$ are positive, m is positive and monotone decreasing and has an integrable singularity at 0. The method used is based on approximating the singular kernel by smooth kernels and showing that it is possible to pass to the limit.

Thus far, we have talked about existence results of local type. This means that we have initial data given at a certain time, and we prove that solutions exist on some time interval thereafter, which may be short. For large times, smooth solutions may not exist as we shall see in the next section. However, there are results on existence of smooth solutions for all times if one considers smooth and small data perturbing a stable state of rest. Dafermos and Nohel [5], [6] have established such a result for the problem

$$u_{tt} = \phi(u_x)_x - \int_{-\infty}^t m(t - \tau) \psi(u_x(\tau))_x d\tau + f(x, t), \quad (1.9)$$

posed on a finite interval with boundary conditions of Dirichlet or Neumann type. The proof is based on energy estimates which guarantee that small solutions will remain small for all times.

Hrusa [13] has generalized this method to a large class of one-dimensional problems. Hrusa and Nohel [14] study the Cauchy problem on all of space for (1.9). This problem is more difficult than the initial-boundary value problem, because an estimate on x -derivatives of a function no longer implies an estimate on the function itself. This makes the required energy estimates far more complicated. In three dimensions, Kim [21] has shown a global existence result on all of space for a covariant quasilinear model fluid. It is to be expected that Kim's proof can be generalized to K-BKZ fluids.

2. PROPAGATION AND DEVELOPMENT OF SINGULARITIES

While parabolic equations, like the heat equation, instantly smooth out discontinuous initial data, hyperbolic equations like the wave equation will propagate discontinuities and, if nonlinearities are introduced, discontinuities can develop from initially smooth data. As we shall see, the intermediate nature of model equations for viscoelastic materials leads to interesting possibilities.

One-dimensional linear wave propagation in a viscoelastic material is described by the equation

$$u_{tt} = bu_{xx} + \mu u_{xxt} + \int_{-\infty}^t m(t - s)(u_{xx}(t) - u_{xx}(s))ds. \quad (2.1)$$

Here b and μ are non-negative constants and m is a non-negative, non-increasing function on $[0, \infty)$ such that m is integrable at ∞ and $tm(t)$ is integrable at 0 .

For the following, we are interested only in the case

$$\mu = 0.$$

In recent joint work with W. Hrusa [16], we study (2.1) for $x \in \mathbb{R}$ with the following initial data

$$u = 0, \quad t < 0$$

$$u(x, t = 0) = \frac{1}{2} \alpha \operatorname{sgn} x, \quad (2.2)$$

$$u_t(x, t = 0) = \frac{1}{2} \beta \operatorname{sgn} x.$$

The problem (2.1), (2.2) can easily be solved by Laplace transforms. We are interested in the regularity of the solution and the location of singularities. It is easy to see that these are determined by the asymptotic behavior at infinity of the Laplace transform of u , which is related to the transform of m .

If m is a smooth function, then, for $\lambda \rightarrow \infty$, $\operatorname{Re} \lambda > 0$, the transform has the asymptotic behavior

$$\hat{m}(\lambda) = \frac{1}{\lambda} m(0) + O\left(\frac{1}{\lambda^2}\right). \quad (2.3)$$

In this case, it is well known [1] that the singularity propagates with the wave speed $\sqrt{b + \hat{m}(0)}$ and decays exponentially with a factor $\exp(-m(0)t/2(b + \hat{m}(0)))$. In addition, we show in [16] that, except in a very special case, there is a discontinuity in u_{xx} across $x = 0$. Away from the lines $x = \pm t\sqrt{b + \hat{m}(0)}$, $x = 0$, the solution is smooth.

The presence of a singularity in m alters the asymptotic behavior of $\hat{m}(\lambda)$. Of particular physical interest are kernels m which are smooth away from 0, but become infinite at 0. Such a possibility has been suggested by certain molecular theories [7], [34], [39] and by experiments [22].

As is expected from the formula for the wave speed above, there remains a finite wave speed as long as m is integrable. However, the exponential decay factor becomes zero if $m(0)$ is infinite, and we can no longer expect

propagation of a discontinuity. In fact, Hannsgen and Wheeler [10] show that, for any singular kernel, there must be some kind of "smoothing"; that is, the solution at any positive time will lie in a more restricted function space than the space admitted for the initial data. How much smoothing there is, depends, not surprisingly, on the nature of the singularity in m . In [16], [28], the following examples are discussed:

$$1. \quad m(t) = \sum_{n=1}^{\infty} e^{-n^{\alpha}t}, \quad \alpha > \frac{1}{2}.$$

This kernel behaves like $t^{-1/\alpha}$ as $t \rightarrow 0$, and is integrable if $\alpha > 1$. The solution is C^{∞} across the wave and analytic behind the wave, i.e. for $0 < x < t \sqrt{b + \hat{m}(0)}$ (the latter is in fact true for any completely monotone kernel). If $\alpha < 1$, m is not integrable. The wave speed is infinite, and the solution is analytic everywhere in $t > 0$, $x \neq 0$.

$$2. \quad m(t) = \sum_{n=1}^{\infty} \exp(-e^n t).$$

This kernel behaves like $|\ln t|$ as $t \rightarrow 0$. The number of derivatives of the solution which exist across the wave is increasing proportional with time.

$$3. \quad m(t) = \int_0^{\infty} \exp(-e^v t) dv.$$

This behaves like $\ln|\ln t|$ as $t \rightarrow 0$. If regularity is measured by the number of existing derivatives, the gain of regularity in the solution is infinitesimal. However, the initial discontinuity disappears and u is continuous across the wave.

We believe that continuity across the wave should hold under very mild assumptions on m (provided of course that m is singular), but we have not been able to prove a general result. It would be enough to show that the total variation remains finite.

As this discussion shows, singular kernels lead to completely new qualitative features of wave propagation. The coexistence of finite speed and smoothing may be of

interest in modeling phenomena in other fields of physics, such as heat flow. It would be extremely interesting to know how singular kernels affect the solutions of nonlinear problems, but no results exist so far. In the following, I shall review some results for nonlinear problems with smooth kernels.

Coleman and Gurtin [3], partly in joint work with Herrera [2] have studied the evolution of acceleration waves. They find that, if an acceleration wave propagates into a medium at rest, then its amplitude satisfies a Riccati equation of the form

$$\frac{da}{dt} = -\gamma a + \frac{\gamma}{\lambda} a^2. \quad (2.4)$$

Here, in our notation from above, $\gamma = \frac{m(0)}{2(b + m(0))}$, and λ is a second constant related to nonlinear properties of the material. From (2.4) it is clear that a will decay exponentially if its initial value is small, but will blow up in finite time if its initial value is large and has the right sign.

This result shows that weak singularities can grow in amplitude and lead to stronger singularities. This, and known results on hyperbolic conservation laws, have motivated the search for results establishing the development of singularities from smooth data. For certain specific model equations, such results were obtained by Slemrod [35], Gripenberg [8], Hattori [9], Malek-Madani and Nohel [24] and Malek-Madani, Nohel and John [25]. We do not present these proofs here, but the following, somewhat sketchy argument will illustrate some of the main ideas. Let us consider the model problem studied by Malek-Madani and Nohel [24]

$$u_t + \phi(u)_x - \int_0^t m(t-s)\psi(u(s))_x ds = 0. \quad (2.5)$$

Here ϕ and ψ are strictly monotone functions:

$\phi', \psi' > 0$ If we set

$$z = - \int_0^t m(t-s)\psi(u(s))ds ,$$

then

$$z_t = -m(0)\psi(u) - \int_0^t m'(t-s)\psi(u(s))ds . \quad (2.7)$$

If t is small and u is bounded, then the integral in (2.7) is small. We can then regard (2.5) as a perturbation of the problem

$$\begin{aligned} u_t + \phi(u)_x + z_x &= 0 , \\ z_t &= -m(0)\psi(u) , \end{aligned} \quad (2.8)$$

and we expect that the solution of (2.5) will develop a discontinuity in u if the solution to (2.8) develops such a discontinuity sufficiently fast, (and in fact, it is not difficult to include the integral as a perturbation in the following argument). In (2.8), we introduce Riemann invariants $r = \phi(u) + z$, $s = z$, and we define the characteristic derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \phi'(u) \frac{\partial}{\partial x}$. We then obtain

$$\begin{aligned} \frac{dr}{dt} &= -m(0)\psi(u) , \\ \frac{\partial s}{\partial t} &= -m(0)\psi(u) . \end{aligned} \quad (2.9)$$

We differentiate with respect to x , and use the fact that $\phi(u) = r - s$, to obtain

$$\begin{aligned} \frac{d}{dt} r_x &= - \frac{\phi''(u)}{\phi'(u)} (r_x - s_x)r_x - m(0) \frac{\psi'(u)}{\phi'(u)} (r_x - s_x) \\ \frac{\partial}{\partial t} s_x &= -m(0) \frac{\psi'(u)}{\phi'(u)} (r_x - s_x) \end{aligned} \quad (2.10)$$

If we now let $\rho = \sup_x |r_x|$, and $\sigma = \sup_x |s_x|$, then, under appropriate hypotheses on ϕ, ψ and the initial data (2.10) leads to differential inequalities of the form

$$\begin{aligned} \left(\frac{d}{dt}\right)_+ \rho &> \alpha \rho^2 - \beta \rho \sigma - \gamma(\rho + \sigma) \\ \left(\frac{d}{dt}\right)_+ \sigma &< \delta(\rho + \sigma) \end{aligned} \quad (2.11)$$

It is not hard to show that we can take data such that ρ tends to ∞ in a short time.

The analogy with hyperbolic conservation laws and the numerical evidence [26] suggest that this blow-up of smooth solutions means the development of shocks (i.e. weak solution with discontinuous u for (2.5) or solutions with discontinuous u_x and u_t for second order equations). No such result has been proved. Existence theorems for solutions with initial data of small total variation were established by Dafermos and Hsiao, but their work is still unpublished.

3. STEADY FLOWS OF VISCOELASTIC FLUIDS

Existence results for steady flows of viscoelastic liquids are lacking, even for slow flows perturbing rest. Formally, such slow flows have been analyzed by perturbation methods. Let the stress be given by a smooth functional of the history of the relative Cauchy strain

$$\underline{T} = \int_{s=-\infty}^t F[\underline{C}_t(s)] \, ds. \quad (3.1)$$

We assume $\underline{C}_t(s)$ is a smooth function of s , expandable in a Taylor series about $s = t$:

$$\underline{C}_t(s) = \underline{1} + \sum_{i=1}^n \frac{(s-t)^i}{i!} \underline{A}_i(t) + O(|s-t|^{n+1}). \quad (3.2)$$

The Rivlin-Ericksen tensors \underline{A}_i satisfy the following recursion relation in a steady flow

$$\begin{aligned} \underline{A}_{i+1} &= (\underline{u} \cdot \nabla) \underline{A}_i + \underline{A}_i \nabla \underline{u} + (\nabla \underline{u})^T \underline{A}_i, \\ \underline{A}_1 &= \nabla \underline{u} + (\nabla \underline{u})^T. \end{aligned} \quad (3.3)$$

We can therefore regard (3.2) as an expansion of $\underline{C}_t(s)$ in increasing powers of a small and smooth velocity field \underline{u} . Suppose, e.g., that we want to solve the

equations of motion

$$\begin{aligned} \rho(\underline{u} \cdot \nabla) \underline{u} &= \operatorname{div} \underline{T} - \nabla p + \varepsilon \underline{f} , \\ \operatorname{div} \underline{u} &= 0 , \\ \underline{u}|_{\partial \Omega} &= \varepsilon \underline{u}_0 , \end{aligned} \quad (3.4)$$

in a bounded domain Ω , where ε is a small parameter. Under appropriate conditions on the functional F (see Coleman and Noll [4]), we can formally expand \underline{u} in powers of ε and equate coefficients of the same powers of ε in the equations. This leads to an infinite sequence of inhomogeneous Stokes problems from which an approximate "solution" can be determined to any order in ε .

It has not been proved that such a formal expansion converges or that it yields an asymptotic approximation to a solution of the full equation. Because of the presence of the term $(\underline{u} \cdot \nabla) \underline{A}_i$ in (3.3), the expansion of $\underline{C}_t(s)$ involves derivatives of \underline{u} of arbitrary order. It is therefore clear that we can expect convergence of the expansion only if we require a high degree of smoothness for the data of the problem. It is possible that, under appropriate assumptions, convergence could be shown in certain spaces of analytic functions. An effort in this direction was made by Niggemann [27]. He studies one-dimensional model problems of the form

$$u'' = f + \sum_{N=1}^{\infty} \sum_{j_1, \dots, j_N=1}^{\infty} a_{j_1 \dots j_N} u^{(j_1)} \dots u^{(j_N)} \quad (3.5)$$

His analysis requires certain assumptions on the coefficients $a_{j_1 \dots j_N}$, for which no physical motivation has been given.

An alternative approach to the problem is the use of iterative schemes, by which it may be possible to prove existence of solutions without recourse to small parameter expansions. The asymptotic validity of such expansions can then be established a posteriori, by estimating derivatives

of the solution with respect to ε . In recent work by the author [33], existence of solutions to (3.4) is proved in this way, provided that \underline{u}_0 is tangential to $\partial\Omega$, and \underline{T} is determined by a differential constitutive law of Maxwell or Jeffreys type, with an arbitrary number of relaxation modes. For simplicity, we shall here consider only the special case of the upper-convected Maxwell model

$$(\underline{u} \cdot \nabla) \underline{T} - (\nabla \underline{u}) \underline{T} - \underline{T} (\nabla \underline{u})^T + \lambda \underline{T} = \eta \lambda (\nabla \underline{u} + (\nabla \underline{u})^T) \quad (3.6)$$

By taking the divergence of this equation, we obtain

$$(\underline{u} \cdot \nabla) \operatorname{div} \underline{T} - (\nabla \underline{u}) \operatorname{div} \underline{T} + \lambda \operatorname{div} \underline{T} = \underline{T} : \partial^2 \underline{u} + \eta \lambda \Delta \underline{u}, \quad (3.7)$$

where $\underline{T} : \partial^2$ stands for $\sum_{j,k} T_{jk} \frac{\partial^2}{\partial x_j \partial x_k}$. We substitute $\operatorname{div} \underline{T} = \rho(\underline{u} \cdot \nabla) \underline{u} + \nabla p - \underline{f}$ from (3.4) (we set $\varepsilon = 1$), introduce the notation $q = (\underline{u} \cdot \nabla) p + \lambda p$, and solve the equations by the following iteration scheme.

$$\underline{u}_0 = \underline{0}, \quad p^0 = q^0 = 0, \quad \underline{T}^0 = \underline{0} \quad (3.8)$$

$$\begin{aligned} \underline{T}^n : \partial^2 \underline{u}^{n+1} + \eta \lambda \Delta \underline{u}^{n+1} - \rho(\underline{u}^n \cdot \nabla)(\underline{u}^n \cdot \nabla) \underline{u}^{n+1} - \nabla q^{n+1} \\ = -[\nabla \underline{u}^n + (\nabla \underline{u}^n)^T] \nabla p^n - (\underline{u}^n \cdot \nabla) \underline{f} + (\nabla \underline{u}^n) \underline{f} - \lambda \underline{f} \\ - \rho(\nabla \underline{u}^n)(\underline{u}^n \cdot \nabla) \underline{u}^n + \lambda \rho(\underline{u}^n \cdot \nabla) \underline{u}^n \end{aligned} \quad (3.9)$$

$$\operatorname{div} \underline{u}^{n+1} = 0, \quad \underline{u}^{n+1}|_{\partial\Omega} = \underline{u}_0, \quad \iiint_{\Omega} q^{n+1} = 0$$

$$(\underline{u}^{n+1} \cdot \nabla) p^{n+1} + \lambda p^{n+1} = q^{n+1} \quad (3.10)$$

$$\begin{aligned} (\underline{u}^{n+1} \cdot \nabla) \underline{T}^{n+1} - (\nabla \underline{u}^{n+1}) \underline{T}^{n+1} - \underline{T}^{n+1} (\nabla \underline{u}^{n+1})^T \\ + \lambda \underline{T}^{n+1} = \eta \lambda (\nabla \underline{u}^{n+1} + (\nabla \underline{u}^{n+1})^T) \end{aligned} \quad (3.11)$$

Equation (3.9) is basically a perturbation of the Stokes problem, while (3.10) and (3.11) are hyperbolic problems which can be solved by the method of characteristics. The proof for convergence of (3.8)-(3.11) is similar to existence proofs for initial value problems for quasilinear

hyperbolic systems. First bounds for all iterates in a sufficiently strong norm are obtained, and then these bounds are used to obtain a contraction estimate in a weaker norm. Iterations similar to (3.8)-(3.11) have been used in numerical calculations (see e.g. [40]).

The following questions remain open:

1. Can similar results be obtained for integral type constitutive models?
2. What happens if \underline{u}_0 is not tangential to $\partial\Omega$? In this case we expect from the analysis of characteristics [18] that extra boundary conditions need to be imposed at inflow boundaries. What are the right conditions to impose and what compatibility conditions are required to avoid singularities of the solution where an inflow boundary joins with other parts of the boundary?
3. Formal expansions for small parameters require no extra boundary conditions at inflow. How are solutions approximated by these expansions distinguished from others?
4. Can one say anything about steady flows when \underline{f} and \underline{u}_0 are not small? In this case it is possible that a change of type occurs in the equations governing steady flow [18]. However, attempts to compute steady flows have encountered difficulties even in situations, such as creeping flows of an upper convected Maxwell model, where there should be no change of type. Although the existence of steady flows in the Newtonian case is well-known [38], even at high Reynolds number, we have no reason to believe that steady flows of viscoelastic fluids should always exist. It is possible that the numerical difficulties result from non-existence.

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